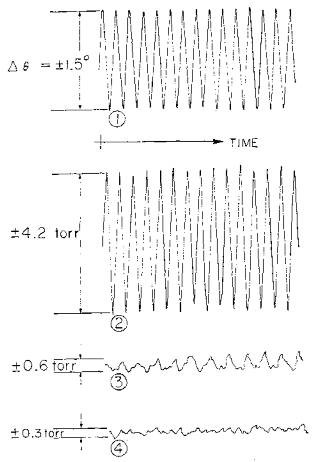

Fig. 2 Sample oscillograph records.

with wind on but no oscillation. Details of the experiments and more complete results are available in Ref. 2.

The experimental (after subtracting noise) and theoretical ratios of the amplitude P_{osc} of the oscillatory pressure to stagnation pressure P_0 at a station 0.93 c behind the wing apex and 0.23 c outboard of the wing root are given in Table 1. The effect of a slight nonzero mean incidence on the measured oscillatory pressure was small, as can be seen by comparing rows 2 and 3 in Table 1. Hence the value in rows 1 and 2 may be compared to the theoretical values which were calculated for zero mean wind incidence. Such theoretical values were obtained by private communication from J. J. Kacprzyński and were based on Ref. 3.

As may be seen, the oscillatory pressure ratio P_{osc}/P_0 for longitudinal oscillation is only of the order of 2–6% of the corresponding value for pitching oscillation. Thus, the effect of wind-tunnel flow discontinuities in this particular case appears to be relatively small. This conclusion is further corroborated by the fact that the results obtained with the model mounted on the sidewall and those obtained with the model mounted on the topwall (i.e., in a completely different local flow environment) agree within 4%.

In cases where the effect of flow discontinuities is not small, the longitudinal oscillation may be used to determine suitable corrections. A recording should be made of the oscillatory pressure for both modes of oscillation vs the motion of the wing. The oscillatory pressure for the pitching oscillation should then be corrected using values obtained for longitudinal oscillation for corresponding locations of the wing station with respect to flow discontinuities. In practice, the frequencies of the two oscillations may be different and, given nearly sinusoidal pressure variations and low reduced frequencies (small phase shifts), the amplitudes of the two oscillatory pressures may be directly superimposed.

It also may be seen from Table 1 that the experimental results, at least for this particular wing station, seem to agree much better with the nonlinear theory than with the linear one. As a further check of the experimental procedure, the measured oscillatory pressure was compared with a quasi-steady prediction based on the measured mean pressure at two different wing incidences. It was found that, within the accuracy of the present measurements, the oscillatory pressure could indeed be considered as quasi-steady, as would be expected for the relatively low reduced frequency of the experiment (k = 0.06). It may be noted that, in general, the small oscillatory pressure component can be measured with higher absolute accuracy than the relatively larger steady pressure component. Thus, a series of pressure measurements on a fixed wing in several successive longitudinal locations would not be as practical as the method described here.

In summary then, the effect of wind-tunnel flow discontinuities, which previously was considered to be a serious obstacle in measuring oscillatory surface pressures at supersonic speeds, was found to be relatively small, at least for the nozzle and for the wind tunnel used for this investigation. An experimental method, employing longitudinal oscillation of the model, was suggested for assessing the magnitude of this effect and for determining, if necessary, the required corrections. The measured oscillatory pressure in the vicinity of the trailing edge of the wing agreed well with the prediction of an unsteady nonlinear supersonic theory.

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Buckling of Clamped Skew Plates

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Nomenclature

a, b	= dimensions of plate, see Fig. 1
D	= $Eh^3/12(1 - \nu^2)$, flexural rigidity of the plate
h	= plate thickness
m, n, r, s	= indices in the deflection series
M	= maximum value of indices m, r
N	= maximum value of indices n, s
N_x, N_{xy}, N_y	= midplane forces per unit length, Fig. 1
$\bar{R}_x, \bar{R}_{xy}, \bar{R}_y$	= nondimensional midplane force parameters, N_xb^2/π^2D, $N_{xy}b^2/\pi^2D$, N_yb^2/π^2D, respectively
$\bar{R}_x^*, \bar{R}_{xy}^*, \bar{R}_y^*$	= $N_xb^2\cos^4\psi/\pi^2D$, $N_{xy}b^2\cos^3\psi/\pi^2D$, $N_yb^2\cos^2\psi/\pi^2D$, respectively
$W(x_1, y_1)$	= transverse deflection of the plate
x, y	= rectangular coordinates, see Fig. 1
x_1, y_1	= oblique coordinates, see Fig. 1
ψ	= angle of skew defined in Fig. 1
ν	= Poisson's ratio

Introduction

In this Note, a condensed version of Ref. 1, only the results are presented. The available results for buckling of clamped skew plates are few and far from complete.[2,3] In the present investigation, results for several new plate configurations and loading conditions as well as more accurate results for configurations reported in previous literature are obtained. In general, for a given a/b, the critical values increase with increasing skew angle. The results also confirm the conjecture of Ref. 4 that in the case of buckling under shear (N_{xy}), two critical values exist, the positive shear (one tending to reduce the skew angle) being numerically greater than the negative shear. However, reliable values for positive shear could not be obtained in Ref. 4 because of convergence difficulties. In

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Table 1 Buckling coefficients under the in-plane force N_x alone

a/b	ψ, deg	M	N	Nature of $(m+n)$	$\bar{R}_{xcr}$	Comparison		
0.5	0	6	6	E[a]	19.35			
	15	4	5	E	21.63			
	30	5	6	E	30.38			
	45	6	6	E	55.26			
	60	6	6	O[a]	130.5			
0.6	0	4	5	E	14.92			
	15	4	5	E	16.49			
	30	4	5	E	22.55		22.52[d]	
	45	4	5	E	39.73		39.92[d]	
	60	6	6	O	90.50			
0.75	0	3	3	E	11.70	11.66[b]		
	15	4	5	E	12.76			
	30	4	6	E	16.71			
	45	6	6	E	27.06			
	60	6	6	E	60.59			
1	0	6	6	E	10.08	10.07[b]		10.15[e]
	15	4	4	E	10.87			
	30	6	6	E	13.58	13.53[c]	13.64[d]	13.76[e]
	45	6	6	E	20.44	20.72[c]	21.64[d]	20.44[e]
	60	6	6	E	42.14			
1.25	0					9.25[b]		
	15	6	4	O	9.915			
	30	5	4	O	12.32		12.44[d]	
	45	5	4	O	18.50		20.56[d]	
	60	6	6	O	38.01			
1.5	0					8.33[b]		8.389[e]
	15	5	4	O	8.971			
	30	5	4	O	11.16		11.34[d]	12.32[e]
	45	5	4	O	17.10		19.40[d]	17.99[e]
	60	6	6	O	36.84			
2	0	4	4	E	8.033	7.88[b]		
	15	4	4	E	8.700			
	30	5	4	E	10.53			
	45	6	5	E	15.74			
	60	6	6	E	39.35			

[a] O = odd; E = even.
[b] See Ref. 14 asympsotic value obtained from 5-, 10-, 15-, 20-, term solution.
[c] See Ref. 7 (11th-order determinant).
[d] See Ref. (8-term result).
[e] See Ref. 12 (144 elements, 415 unknowns).

Table 2 Buckling coefficients under the in-plane force N_{xy} alone

a/b	ψ, deg	M	N	Nature of $(m+n)$	$\bar{R}_{xycr}$	Comparison		
0.5	0						±42.28[c]	
	15	4	4	E[a]	−34.58			
				O[a]	55.36			
	30	4	5	E	−31.58			
				E	76.90			
	45	6	6	E	−40.54			
				O	128.3			
	60	6	6	E	−85.00			
				O	531.5			
1	0						±14.83[c]	±14.88[d]
	15	4	4	E	−14.39			
				E	17.24			
	30	5	4	E	−16.66			−16.69[d]
				E	23.64			
	45	6	6	E	−24.08	−24.32[b]		−24.41[d]
				O	+32.56			
	60	6	6	O	−46.58			
				O	69.86			
1.5	0						±11.56[c]	±11.67[d]
	15	6	6	O	−12.01			
		4	3	E	12.73			
	30	6	6	O	−14.05			−14.12[d]
				E	15.19			
	45	6	6	O	−20.21			−20.64[d]
				E	22.37			
	60	6	6	E	−40.24			
				E	45.83			
2	0						±10.57[c]	
	15	6	6	O	−10.84			
				O	11.10			
	30	6	6	E	−13.34			
				O	13.73			
	45	6	6	E	−19.24			
				O	20.35			
	60	6	6	E	−39.38			
				O	44.40			

[a] E = even; O = odd.
[b] See Ref. 4 (8-term result).
[c] See Ref. 3 (Royal Aeronautical Society data sheets).
[d] See Ref. 12 (144 elements, 415 unknowns).

our investigation, we have been able to obtain the positive critical value also satisfactorily.[5] Further, as may be expected, the magnitude of critical values is higher than that of corresponding simply supported skew plates.[6] Also, it is felt that the use of bar buckling eigenfunctions for the solution of plate buckling problems should prove as attractive as the use of beam characteristic functions for plate vibration problems and this aspect needs to be explored further. With the analysis and the computer program developed in this paper, it is possible to calculate and obtain the interaction surfaces for buckling under combined loads which will be useful for design purposes.

Brief Description of Solution

References 7–12 considered different aspects of the buckling problem of clamped skew plates and presented a few results each for some configurations. The details of these investigations will have to be omitted due to exigencies of limited space. In this Note, the buckling problem of clamped skew plates under any uniform system of applied stresses is investigated using the Galerkin method, expressing the deflection as a double series of beam characteristic functions[13] in terms of oblique coordinates. The boundary conditions of zero deflection and normal slope are satisfied on all the edges by these functions. The in-plane forces are represented in terms of orthogonal components. Although it may have been more relevant to represent the stress system in terms of oblique components, as the plate under consideration is skew, the orthogonal components are preferred since the stresses in this system retain their true physical significance. The transformation between the two systems is, of course, well known.[2] The application of the Galerkin method finally results in a matrix equation which splits into two sets, one in which both $(m + n)$ and $(r + s)$ are even and the other in which both $(m + n)$ and $(r + s)$ are odd. These are labeled the even (E) set and the odd (O) set, representing groups of modes which are skew symmetric and skew antisymmetric, respectively (see Fig. 1). The buckling parameter, obviously, is the lower of the two lowest eigenvalues obtained from the two distinct cases even and odd. The eigenvalues for a given configuration of the plate when N_x, N_{xy}, N_y are present individually or in combination are calculated by appropriately assigning numerical values to any two of the three parameters $\bar{R}_x^*$, $\bar{R}_{xy}^*$, $\bar{R}_y^*$ and treating the third one as the eigenvalue to be determined.

Results and Discussion

Numerical calculations have been performed for different combinations of side-ratio a/b and angle of skew ψ. Convergence naturally tends to deteriorate with increasing angle of skew. Thus, rather judiciously, smaller number of terms were used at low angles of skew and larger number of terms at higher angles of skew. The number of terms were limited to $M = 6$, $N = 6$, which results in matrices of maximum order of 18×18 each for the even case and the odd case. Most of the

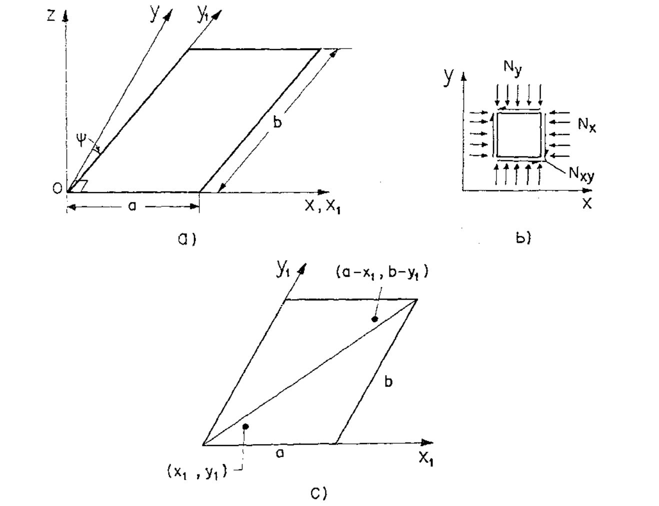

Fig. 1 Sketch of the skew panel, showing the coordinate axes, the in-plane force system, and the symmetries of the skew domain. $W(a - x_1, b - y_1) = W(x_1, y_1) \sim$ skew symmetric; $W(a - x_1, b - y_1) = -W(x_1, y_1) \sim$ skew antisymmetric.

calculations performed pertained to the case of buckling under the action of the forces N_x, N_{xy}, N_y individually and a few under the combined action of two out of the three. The results are presented in Tables 1–4 along with such of those results which are available. From the results of the convergence study,[1] it could be concluded that whereas the convergence with 18 terms appeared to be quite adequate in the case of buckling under N_x and N_y, it appeared to become slower in the case of N_{xy}. The need for the number of terms of the order 15–18 is not uncommon even for rectangular plates.[14] In any case, although the results for $\psi = 60°$ are reported for completeness, they would have to be used with some caution. The convergence problems, naturally, are far less severe at smaller skew angles and correspondingly smaller number of terms can be, and in fact have been, used as may be seen from the tables. In comparison with the results reported by Argyris,[12] it is clear that continuum approximate methods for uniform plates are decidedly preferable to the matrix displacement methods which necessarily involve handling of very large order matrices. It is felt that the use, in Galerkin method, of products of buckling eigenfunctions of bars with end conditions appropriate to the plate edge conditions, may prove advantageous in treating the buckling problems of skew plates.

From Table 1, it can be seen that the buckling coefficients $\bar{R}_{xcr}$ obtained are better than the values reported by Wittrick,[8] particularly, at the higher angles of skew and for $a/b \geqslant 1$. Probably this is due, primarily, to more number of terms used in the present analysis. Further, our values are better than the values reported by Argyris[12] using the matrix displacement method which involved working with 415 unknowns. Guest[7] claimed to have obtained lower bounds in the case of $a/b = 1$, $\psi = 30°$ and 45° but his so-called lower bound for $\psi = 45°$ is 1.4% higher than the 18-term result of the present paper which is known to be definitely an upper bound.

Wittrick[4] and Argyris[12] reported the results for negative shear only (Table 2). Again, it can be seen that the results of the present paper are better than the results of Refs. 4 and 12. Further comparison with their results is not possible as the configurations of the skew plate considered are different.

In Table 3, the results for the buckling coefficients under N_y are presented. In Table 4, the results for the buckling coefficient $\bar{R}_{ycr}$ under the combined action of N_y and N_{xy} (positive and negative) for a few cases are presented. Comparison of our results with the results of Refs. 9 and 11 is not possible as the loadings considered do not correspond with each other.

Table 3 Buckling coefficients under the in-plane force N_y alone

a/b	ψ, deg	M	N	Nature of $(m + n)$	$\bar{R}_{ycr}$
	0	4	4	E^a	32.13
	15	4	4	E	34.09
0.5	30	4	4	O^a	39.72
	45	6	6	O	53.22
	60	6	6	E	86.20
	0	6	6	E	10.08
	15	6	6	E	10.43
1	30	6	6	E	11.76
	45	6	6	E	15.26
	60	6	6	E	25.78
	0				5.78^b
	15	4	4	E	6.151
1.5	30	6	4	E	7.271
	45	6	6	E	10.10
	60	6	6	E	18.54
	0	6	6	E	4.838
	15	6	6	E	5.132
2	30	6	6	E	6.208
	45	6	6	E	8.938
	60	6	6	E	17.08

^{a} E = even; O = odd.
^{b} By interpolation from results of Table 1.

Table 4 Buckling coefficients under a few combined loadings

a/b	ψ, deg	$\bar{R}_x$*	$\bar{R}_{xy}$*	$\bar{R}_y$*	M	N	Nature of $(m+n)$	$\bar{R}_{cr}$
	15	0	8	...	4	4	E^a	33.74
		0	−8	...	4	4	O^a	31.00
		0	12	...	4	4	E	32.75
0.5		0	−12	...	4	4	O	28.30
	30	0	8	...	4	5	E	39.37
		0	−8	...	4	5	O	33.40

^{a} E = even; O = odd.

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A Use of Spline Data Techniques with Optimization Programs

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OVER the past few years a great deal of interest and effort has been directed towards the optimization of trajectories. However, in using some of the optimization schemes numerical difficulties have been encountered which, at times, seem to be problem dependent; problems involving only a small portion of flight time in the atmosphere have generally been successfully solved, whereas problems entailing increased portions of atmospheric flight time have tended to have less likelihood of success. One reason for the additional difficulties in atmospheric flight is believed to stem from the way in which aerodynamic data curves are usually input to the problem. This Note discusses one way in which errors are introduced into such problems and offers one technique for the elimination of such errors.

A widely used method of curve representation consists of simple tabular input data with linear interpolation used for evaluation between the data points. For example, in the aircraft flight differential equations of Eq. (1), input data are used to represent thrust, T, mass flow rate, $\dot{m}$, and the lift and drag coefficients (C_L and C_D, respectively);

$$\dot{v} = (T/m)\cos\alpha - \rho(V^2S/2m)C_D - g\sin\gamma$$
$$\dot{\gamma} = \rho(VS/2m)C_L + (T/mV)\sin\alpha - g\cos\gamma/V \qquad (1)$$
$$\dot{h} = V\sin\gamma,\ \dot{x} = V\cos\gamma,\ \dot{m} = \dot{m}(h,M)$$

or, in general vector form

$$\dot{x} = f(x,\alpha,t)$$

However, in using the simple tabular input form of data curve representation, it is important to realize that, at the data points, the slope of the represented curve is discontinuous. When an optimization problem is attempted using either a direct or indirect method, these discontinuities play an important role as, with both methods, numerical integration of an auxiliary set of differential equations of the vector form

$$\dot{\lambda} = -f_x{}^T\lambda \qquad (2)$$

where

$$\dot{x} = f(x,\alpha,t)$$

is required. In both methods, the discontinuities of the slope of the input data curves are reflected as jump discontinuities on the right side of Eq. (2) in the auxiliary variables. Recognition of this error source makes it desirable to find a means of data curve representation that exhibits continuous first derivatives.

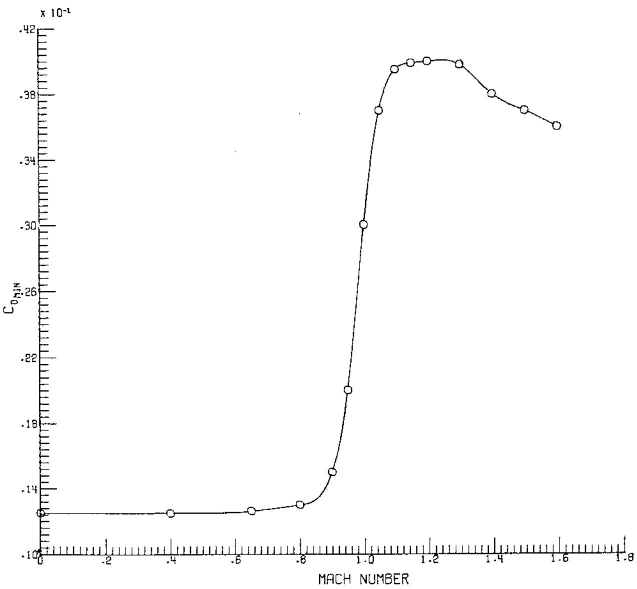

Fig. 1 Example of spline curve fit to a set of input points.

Several methods of curve fittings are now in general use. Recently the spline function has been generating considerable interest. See for example Refs. 1–3. In particular the cubic spline function appears to be quite useful since in certain applications the cubic spline minimizes the integral of the curvature over the entire function.[4] By construction each segment of the function, i.e., between input data points, is a cubic polynominal, such that adjoining cubics and their first two derivatives are continuous. Thus the cubic spline representation of input data curves represent one way of eliminating the difficulties resulting from the discontinuous first derivatives.

To evaluate the benefits possible through the use of such a data representation, a simple minimum time-to-climb aircraft flight-path problem was set up. The aircraft model chosen was one illustrative of a supersonic interceptor. In representing the input data curves, tabular data points were picked off the plots so as to provide for a reasonable fit to the data curves. As an example of the representation provided by the tabular input data format, Fig. 1 shows a choice of data points which might be used to represent the minimum drag coefficient, $C_{D_{\min}}$, as a function of Mach number. In this case, the input data points are shown as the symboled points, and straight line interpolation is then used between the points. Such a representation is normally in use with most optimization programs. However, using the same data points, a cubic spline fit may be made through the points using a spline routine similar to the one described in Ref. 4. Such a fit is shown as the solid line of Fig. 1 and may be seen to have a continuous first derivative. Cubic spline fits were also made to the thrust and mass flow data and other aerodynamic data in preparation for use with a digitaloptimiization program.

The problem solution was attempted with two different computer programs employing different optimization schemes. One of the programs used a calculus of variations approach to solve the problem. In using this method, one of the necessary conditions for a calculated trajectory to be optimal is that the Hamiltonian of the system, computed by the matrix equation

$$H = p^T\dot{x} - 1 \qquad (3)$$

where

$$\dot{p} = -f_x{}^Tp$$

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